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On oscillatory solutions of quasilinear differential equations

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Abstract

Necessary and sufficient conditions for the existence of at least one oscillatory solution of a second-order quasilinear differential equation are presented. These results yield also new conditions guaranteeing the coexistence of oscillatory and nonoscillatory solutions. Our approach is based on the asymptotic representation of solutions by means of a periodic function and of a suitable zero-counting function.

Keywords: Second-order quasilinear differential equation; Oscillatory solution; Nonoscillatory solution; Asymptotic representation

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1. Introduction

Consider the quasilinear differential equation

$$\big(a(t)\Phi_p(x')\big)' + b(t)\Phi_q(x) = 0, \quad t \in \mathbb{R}_+ = [0,\infty), \tag{1}$$

where a, b are continuous and positive functions on $\mathbb{R}_+$, $\Phi_p(u) = |u|^{p-2}u$, $p > 1$, $\Phi_q(u) = |u|^{q-2}u$, $q > 1$, and the function $a^{1/(p-1)}b$ is continuously differentiable on $\mathbb{R}_+$. Under these conditions, every solution of (1) can be continued to the whole $\mathbb{R}_+$ (see, e.g., [9]). A solution x of (1) is said to be *oscillatory* if it has arbitrarily large zeros, i.e., if there exists $\tau_n \to \infty$ with $x(\tau_n) = 0$. Otherwise, the solution x is said to be *nonoscillatory*, and (1) is called *oscillatory* or *nonoscillatory* according to every nontrivial solution of (1) is oscillatory or nonoscillatory, respectively.

Equation (1) includes the Emden–Fowler equation

$$\big(a(t)x'\big)' + b(t)|x|^{\mu}\operatorname{sgn} x = 0, \quad \mu \neq 1. \tag{2}$$

The oscillation of (1) and (2) has been extensively studied. We refer to the papers [2,10], the monographs [1,8,9] and references therein. Other contributions can be obtained from [11,12] in which more general equations including (1) are considered.

Although results concerning existence of nonoscillatory solutions and criteria which guarantee that all solutions of (1) are oscillatory are easily accessible in the literature, conditions under which oscillatory and nonoscillatory solutions of (1) may coexist or conditions under which (1) is nonoscillatory are more difficult to obtain.

Our aim is to give necessary and sufficient conditions for the existence of at least one oscillatory solution of (1). These conditions yield new conditions guaranteeing the coexistence of oscillatory and nonoscillatory solutions, as well as the nonoscillation of (1).

A classical approach for the existence of at least one oscillatory solution is due to Kurzweil and Jasný and is based on certain properties of an auxiliary energy-type function. Such an approach has been generalized in various directions: we refer to [2,9,10] and references therein for more details. Our approach used here is different to this one. It is based on the asymptotic representation of any solution x of (1) and its quasiderivative $x^{[1]}$, where

$$x^{[1]}(t) = a(t)\Phi_p\big(x'(t)\big).$$

This method originates to Kiguradze [8] for (2) and later on has been extended by Chanturia [6] and Mirzov [9, §17] for Emden–Fowler type systems. The paper is organized as follows. Section 2 is devoted to the asymptotic representation and properties of a zero-counting function φ, which is associated to any solution of (1). In Section 3 we study the existence of oscillatory solutions of (1). These results extend analogous ones in [9, §13], stated for an Emden–Fowler type system. Finally, in Section 4 we consider the coexistence of oscillatory and nonoscillatory solutions, jointly with a comparison with known results.

We close with some notations. Put

$$\alpha = \frac{p}{pq-q+p}, \qquad \beta = \frac{(p-1)q}{pq-q+p}, \qquad \gamma = \frac{p}{(p-1)q}, \qquad \delta = \frac{p}{p-1},$$
$$\beta_1 = \frac{p-1}{pq-q+p}, \qquad \beta_2 = \frac{q-p}{pq}, \qquad B(t) = a^{\frac{1}{p-1}}(t)b(t). \tag{3}$$

Observe that $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta = 1$. For any solution x of (1), define the function

$$F(t) = B^{\beta}(t)\left(\frac{|x^{[1]}(t)|^{\delta}}{B(t)} + \gamma\left|x(t)\right|^{q}\right). \tag{4}$$

If x is nontrivial, then all its zeros are simple, there is no finite accumulation point of them (see, e.g., [3]) and hence F is positive on $\mathbb{R}_+$.

2. Asymptotic representation

In this section, following the Mirzov approach [9, §17], we give an asymptotic representation of solutions of (1) by means of a periodic function w and of a suitable "zero-counting" function φ. The function φ is a solution of a certain first-order differential equation and, in some sense, it places the role of Prüfer transformation in the linear case.

Lemma 1. *The equation*

$$\big(\Phi_p(w')\big)' + \Phi_q(w) = 0 \tag{5}$$

has a periodic solution w fulfilling $w(0) = c$, $w'(0) = 0$, where c is a real nonzero constant. If $2T$ is the smallest positive period of w, then $w(0) = -w(T)$. Moreover,

w is decreasing on $[0, T]$ and increasing on $[T, 2T]$

or

w is increasing on $[0, T]$ and decreasing on $[T, 2T]$,

according to $c > 0$ or $c < 0$, respectively.

Proof. The assertion can be obtained, e.g., from results of Drábek and Manashevich [7] with some modifications. Here, for sake of completeness, we present a direct proof, based on a recursive approach.

Let y_0 be a solution of (5) satisfying $y_0(0) = c > 0$, $y_0'(0) = 0$. Since (5) is oscillatory (see, e.g., [9]), there exists $T > 0$ such that $y_0(T/2) = 0$, $y_0(t) > 0$ for $t \in (0, T/2)$, i.e., $T/2$ is the smallest zero of y. Clearly y_0 is decreasing on $[0, T/2]$. Consider the function

$$y_1(t) = -y_0(T - t), \quad t \in [T/2, T].$$

It is easy to verify that y_1 satisfies (5) on $[T/2, T]$ and

$$y_1(T/2) = y_0(T/2) = 0, \quad y_1'(T/2) = y_0'(T/2), \quad y_1(t) < 0 \quad \text{for } t \in (T/2, T].$$

Clearly y_1 is decreasing on $[T/2, T]$ and $y_1(T) = -c$. Denote with $\bar{w}$ the solution of (5) given by $\bar{w}(t) = y_0(t)$ on $[0, T/2]$ and $\bar{w}(t) = y_1(t)$ on $[T/2, T]$. Hence $\bar{w}(t)$ is decreasing on $[0, T]$ and $\bar{w}(T) = -c$. Now consider the function

$$y_2(t) = \bar{w}(2T - t), \quad t \in [T, 2T].$$

By the same argument we obtain that y_2 satisfies (5) on $[T, 2T]$ and

$$y_2(T) = \bar{w}(T), \qquad y_2'(T) = -\bar{w}'(T) = 0.$$

Hence, denote with w the solution of (5) on $[0, 2T]$ given by $w(t) = \bar{w}(t)$ on $[0, T]$ and $w(t) = y_2(t)$ on $[T, 2T]$. Clearly $w(0) = w(2T) = c$, $w'(0) = w'(2T) = 0$. Repeating this argument, we can extend the solution w to $\mathbb{R}_+$ in such a way that w is a $2T$-periodic function $\mathbb{R}_+$ fulfilling $w(0) = -w(T)$, decreasing or increasing, according to $t \in [0, T]$ or $t \in [T, 2T]$, respectively. Using a similar argument, with minor changes, the assertion can be proved also in case $c < 0$. The details are left to the reader. □

Lemma 2. *For every nontrivial solution x of* (1) *there exists a function φ, continuously differentiable on $\mathbb{R}_+$, such that*

$$x(t) = B^{-\beta_1}(t)F^{1/q}(t)w\big(\varphi(t)\big),$$
$$x^{[1]}(t) = B^{\beta_1}(t)F^{1/\delta}(t)\Phi_p\big(w'\big(\varphi(t)\big)\big), \tag{6}$$

where w is the periodic solution of (5) *defined in Lemma* 1 *with $c = \gamma^{-1/q}$. The function φ is positive for $t > 0$ and satisfies*

$$\varphi'(t) = a^{-\frac{1}{p-1}}(t)B^{\alpha}(t)F^{\beta_2}(t) + \frac{1}{q}\frac{B'(t)}{B(t)}w\big(\varphi(t)\big)\Phi_p\big(w'\big(\varphi(t)\big)\big). \tag{7}$$

Proof. Let $\{\tau_n\}$ be the sequence (finite or infinite) of all zeros of x' on $\mathbb{R}_+$. Put

$$h(t) = x(t)B^{\beta_1}(t)F^{-1/q}(t), \quad t \in \mathbb{R}_+.$$

According to (4), we obtain $F(t)B^{-\beta}(t) > \gamma|x(t)|^q$ or $F^{1/q}(t)B^{-\beta_1}(t) > \gamma^{1/q}|x(t)|$ for $t \neq \tau_n$. Hence we have for $t \neq \tau_n$,

$$\big|h(t)\big| < \gamma^{-1/q}. \tag{8}$$

Clearly (8) holds for any $t \in \mathbb{R}_+$, if x' does not have zeros on $\mathbb{R}_+$. Similarly we obtain

$$h(\tau_n) = \gamma^{-1/q} \operatorname{sgn} x(\tau_n). \tag{9}$$

Now we construct the function φ satisfying (6). Consider the following cases:

(c_1) x' does not have zeros on $\mathbb{R}_+$;
(c_2) x' has a finite number N of zeros on $\mathbb{R}_+$, $N > 0$;
(c_3) x' has infinitely many zeros on $\mathbb{R}_+$.

In case (c_1), in view of (8), we define

$$\varphi(t) = w_0\big(x(t)B^{\beta_1}(t)F^{-1/q}(t)\big), \quad t \in \mathbb{R}_+,$$

where w_0 is the inverse function to w on $[0, T]$ (on $[T, 2T]$) if $x'(0) < 0$ (if $x'(0) > 0$). Hence φ is positive on $\mathbb{R}_+$.

Consider cases (c_2), (c_3). First we suppose $x(\tau_1) < 0$. If $\tau_1 > 0$ define

$$\varphi(t) = w_0\big(h(t)\big), \quad t \in [0, \tau_1),$$

and for $t \geqslant \tau_1 \geqslant 0$ define

$$\varphi(t) = w_n\big(h(t)\big) \quad \text{on } t \in [\tau_n, \tau_{n+1}), \tag{10}$$

where w_n is the inverse function of w on the interval $[nT, (n+1)T)$. In case (c_2) put $\tau_{N+1} = \infty$. Hence φ is defined on $\mathbb{R}_+$. Because $w_n(\gamma^{-1/q} \operatorname{sgn} x(\tau_n)) = nT = \varphi(\tau_n)$, we have

$$\lim_{t\to\tau_n-} \varphi(t) = \lim_{t\to\tau_n-} w_{n-1}\big(h(t)\big) = nT = \varphi(\tau_n),$$

and so φ is continuous on $\mathbb{R}_+$ and differentiable for $t \neq \tau_n$. In view of the monotonicity of w_n, we obtain

$$nT < \varphi(t) < (n+1)T \quad \text{for } t \in (\tau_n, \tau_{n+1}). \tag{11}$$

From here and (10) the function φ is positive on $[\tau_n, \tau_{n+1})$. If $x(\tau_1) > 0$, we proceed in a similar way: in this case w_n denotes the inverse function of w on the interval $[(n+1)T, (n+2)T)$ and the inequality (11) takes the form

$$(n+1)T < \varphi(t) < (n+2)T \quad \text{for } t \in (\tau_n, \tau_{n+1}). \tag{12}$$

Summarizing, the function φ is positive on $\mathbb{R}_+$, $w(\varphi(t)) = h(t)$ and

$$\operatorname{sgn} x^{[1]}(t) = \operatorname{sgn} w'\big(\varphi(t)\big) = \operatorname{sgn} \Phi_p\big(w'\big(\varphi(t)\big)\big). \tag{13}$$

Hence, the first equality in (6) holds. Since

$$\big(\Phi_p(w')\big)' + \Phi_q(w) = \frac{d}{dt}\big(\big|\Phi_p(w')\big|^\delta + \gamma\big|w(t)\big|^q\big), \quad \text{a.e. on } \mathbb{R}_+,$$

from (5), taking into account $c = \gamma^{-1/q}$, we obtain for $t \in \mathbb{R}_+$,

$$\big|\Phi_p(w')\big|^\delta + \gamma\big|w(t)\big|^q = 1. \tag{14}$$

Further, in view of (4), (6) and (14), we have on $\mathbb{R}_+$

$$F(t) - \big|x^{[1]}(t)\big|^\delta B^{\beta-1}(t) = \gamma F(t)\big|w\big(\varphi(t)\big)\big|^q = F(t)\big(1 - \big|\Phi_p\big(w'\big(\varphi(t)\big)\big)\big|^\delta\big),$$

or

$$\big|x^{[1]}(t)\big| = B^{\beta_1}(t)F^{1/\delta}(t)\big|\Phi_p\big(w'\big(\varphi(t)\big)\big)\big|.$$

From this and from (13), the second equality in (6) follows. Now, we estimate φ. Denote $\rho(t) = F^{1/q}(t)$, $t \in \mathbb{R}_+$. The first equation in (6) yields

$$x'(t) = \big(B^{-\beta_1}(t)\rho(t)\big)'w\big(\varphi(t)\big) + B^{-\beta_1}(t)\rho(t)w'\big(\varphi(t)\big)\varphi'(t)$$

for all $t \neq \tau_n$. From this and from the second equation in (6) we have

$$\begin{aligned}
&a^{-\frac{1}{p-1}}(t)B^{\alpha/p}(t)\rho^{q/p}(t)\big|\Phi_p\big(w'\big(\varphi(t)\big)\big)\big|^\delta \\
&\quad = a^{-\frac{1}{p-1}}(t)\big|x^{[1]}(t)\big|^{\frac{1}{p-1}}\big|\Phi_p\big(w'\big(\varphi(t)\big)\big)\big| \\
&\quad = \big|x'(t)\Phi_p\big(w'\big(\varphi(t)\big)\big)\big| = x'(t)\Phi_p\big(w'\big(\varphi(t)\big)\big) \\
&\quad = \big(B^{-\beta_1}(t)\big)'\rho(t)w\big(\varphi(t)\big)\Phi_p\big(w'\big(\varphi(t)\big)\big) + B^{-\beta_1}(t)\rho'(t)w\big(\varphi(t)\big)\Phi_p\big(w'\big(\varphi(t)\big)\big) \\
&\qquad + B^{-\beta_1}(t)\rho(t)\big|\Phi_p\big(w'\big(\varphi(t)\big)\big)\big|^\delta\varphi'(t);
\end{aligned}$$

hence

$$\begin{aligned}\big|\Phi_p\big(w'(\varphi(t))\big)\big|^{\delta}\varphi'(t) &= a^{-\frac{1}{p-1}}(t)B^{\alpha}(t)\rho^{\frac{q-p}{p}}(t)\big|\Phi_p\big(w'(\varphi(t))\big)\big|^{\delta}\\ &\quad+\left[\beta_1\frac{B'(t)}{B(t)}-\frac{\rho'(t)}{\rho(t)}\right]w\big(\varphi(t)\big)\Phi_p\big(w'(\varphi(t))\big).\end{aligned}\tag{15}$$

Similarly, (1) and (6) yield

$$\begin{aligned}&-b(t)B^{-\beta_1(q-1)}(t)\rho^{q-1}(t)\big|\big(w(\varphi(t))\big)\big|^{q-1}\operatorname{sgn}x(t)\\ &\quad=-b(t)\Phi_q\big(x(t)\big)=\big(x^{[1]}(t)\big)'\\ &\quad=\big(B^{\beta_1}(t)\rho^{1/\gamma}(t)\big)'\Phi_p\big(w'(\varphi(t))\big)-B^{\beta_1}(t)\rho^{1/\gamma}(t)\Phi_q\big(w(\varphi(t))\big)\varphi'(t).\end{aligned}$$

From this, multiplying by $\gamma w(\varphi(t))$ and dividing by $B^{\beta_1}(t)\rho^{1/\gamma}(t)$, we obtain

$$\begin{aligned}&\gamma\big|w(t)\big|^{q}\varphi'(t)\\ &\quad=\gamma b(t)B^{-\beta}(t)\rho^{\frac{q-p}{p}}\big|w\big(\varphi(t)\big)\big|^{q}+\left[\beta_1\gamma\frac{B'(t)}{B(t)}+\frac{\rho'(t)}{\rho(t)}\right]w\big(\varphi(t)\big)\Phi_p\big(w'(\varphi(t))\big).\end{aligned}$$

Now, by adding (15) and using (14) and $a^{-\frac{1}{p-1}}(t)B^{\alpha}(t)=b(t)B^{-\beta}(t)$, we obtain the equality (7) for $t\neq\tau_n$. Using the Mean Value Theorem, we obtain

$$\varphi'_-(\tau_n)=\lim_{t\to\tau_n-}\frac{\varphi(t)-\varphi(\tau_n)}{t-\tau_n}=\lim_{t\to\tau_n-}\varphi'(\xi_n)=a^{-\frac{1}{p-1}}(\tau_n)B^{\alpha}(\tau_n)F^{\beta_2}(\tau_n)=\lambda_n,$$

where $\xi_n\in(t,\tau_n)$. Analogously, a similar argument yields $\varphi'_+(\tau_n)=\lambda_n$. Hence φ is differentiable on the whole $\mathbb{R}_+$ and (7) holds on $\mathbb{R}_+$. □

Remark 1. In case $p\neq q$ the function φ depends on the choice of the solution x and Theorem 1 states that for every solution such function exists.

If $p=q$, then (1) reduces to the half-linear equation

$$\big(a(t)\Phi_p(x')\big)'+b(t)\Phi_p(x)=0.\tag{16}$$

In this case, $\alpha=1/p$, $\beta_2=0$ and φ is a solution of the equation

$$\varphi'(t)=a^{-1/p}(t)b^{1/p}(t)+p^{-1}B'(t)B^{-1}(t)w\big(\varphi(t)\big)\Phi_p\big(w'(\varphi(t))\big),$$

i.e., φ does not depend on the choice of the solution x of (16).

The function φ plays a crucial role in the oscillatory behavior of solutions of (1), as the following result illustrates.

Theorem 1. *Any nontrivial solution x of* (1) *is oscillatory if and only if* $\lim_{t\to\infty}\varphi(t)=\infty$*, where φ is defined in Lemma* 2.

Proof. If x is oscillatory, then the sequence $\{\tau_n\}$ of zeros of x' is infinite. Now the assertion follows from (11) or (12), as $n\to\infty$, according to $x(\tau_1)$ is negative or positive, respectively. Conversely the assertion follows from (6), by taking into account that the solution w of (5) is oscillatory. □

From here and Remark 1 we obtain the well-known property that the half-linear equation (16) is either oscillatory or nonoscillatory.

3. Existence of oscillatory solutions

In this section, we study the existence of at least one oscillatory solution of (1). We will make use of the following hypothesis:

$$g(t) = -\frac{a^{\frac{1}{p-1}}(t)B'(t)}{B^{\alpha+1}(t)} \quad \text{is of bounded variation and } g \in AC_{\mathrm{loc}}(\mathbb{R}_+). \tag{17}$$

Since g is of the bounded variation, there exists $\lim_{t\to\infty} g(t) = \ell_g$ ($|\ell_g| < \infty$), and so g is bounded on $\mathbb{R}_+$. Denote

$$M_1 = \gamma^{-1/q} \sup_{t\in\mathbb{R}_+} \left|g(t)\right|. \tag{18}$$

In the sequel, we will describe the behavior of the function F defined in (4), which will play a crucial role in our later consideration.

Lemma 3. [4, Lemma 1] *Assume* (17). *Then for any solution x of* (1)*, we have*

$$\left|x(t)x^{[1]}(t)\right| \leqslant \gamma^{-1/q} F^{\beta_3}(t), \quad t \in \mathbb{R}_+,$$

where $\beta_3 = (pq - q + p)/(pq)$. In addition for the function F defined in (4)*, we have for $0 \leqslant \tau < t$,*

$$F(t) = F(\tau) - \alpha g(\tau)x(\tau)x^{[1]}(\tau) + \alpha g(t)x(t)x^{[1]}(t) - \alpha \int_\tau^t g'(\sigma)x(\sigma)x^{[1]}(\sigma)\,d\sigma.$$

Lemma 4. *Assume* (17).

(a) *If $q > p$, then for every solution x of* (1) *the function F, given in* (4)*, is bounded on $\mathbb{R}_+$ and there exists a positive constant c and a solution x such that $F(t) \geqslant c$.*
(b) *If $q < p$, then for every nontrivial solution x of* (1) *the function F, given in* (4)*, is bounded away from zero on $\mathbb{R}_+$ and there exists a positive constant c and a solution x such that $F(t) \leqslant c$.*

Moreover, in both cases c can be chosen arbitrarily large or small according to $q > p$ or $q < p$, respectively.

Proof. Claim (a) follows from [4, Lemmas 2, 3, 7].

Claim (b). Assume that there exists a nontrivial solution x of (1) such that

$$\liminf_{t\to\infty} F(t) = 0. \tag{19}$$

Then for any $t_0 \in \mathbb{R}_+$ there exist τ and σ such that $t_0 \leqslant \sigma < \tau$,

$$2F(\tau) = F(\sigma) = F(t_0), \quad F(\tau) \leqslant F(t) \leqslant F(\sigma) \quad \text{for } \sigma \leqslant t \leqslant \tau. \tag{20}$$

Then by Lemma 3, we have

$$\frac{F(t_0)}{2} = F(\sigma) - F(\tau) \leqslant \bar{\gamma} F^{\beta_3}(\sigma)\left(\left|g(\sigma)\right| + \left|g(\tau)\right| + \int_\sigma^\tau \left|g'(s)\right| ds\right)$$
$$\leqslant \bar{\gamma} N F^{\beta_3}(\sigma) = \bar{\gamma} N F^{\beta_3}(t_0),$$

where $\bar{\gamma} = \alpha\gamma^{-1/q}$, and N is a suitable large constant. Hence, using (20) we have $(F(t_0))^{(q-p)/pq} \leqslant 2\bar{\gamma}N$ or, because $q < p$, $F(t_0) \geqslant (2\bar{\gamma}N)^{(q-p)/pq}$.

From (20), we have $F(t_0) \leqslant 2F(t) \leqslant 2F(t_0)$ for $t_0 \leqslant t \leqslant \tau$ and, because t_0 can be chosen arbitrary large, we obtain a contradiction with (19). The remaining part of claim (b) follows again from [4, Lemmas 2, 3]. □

When (17) holds, then $\lim_{t\to\infty} g(t) = \ell_g$, where $\ell_g \in \mathbb{R}$. Hence Lemmas 2 and 4 generalize the asymptotic formulae (17.43) given in [9, Theorem 17.7], where $\lim_{t\to\infty} g(t) = 0$ is assumed and only solutions for which $\lim_{t\to\infty} F(t) = c$ $(0 < c < \infty)$ are considered.

Lemma 5. *Assume* (17) *and* $p \neq q$. *Then* (1) *has an oscillatory solution* x *if and only if*

$$\int_0^\infty a^{-\frac{1}{p-1}}(t) B^\alpha(t)\, dt = \int_0^\infty \left(a^{-\frac{1}{p-1}}(t)\right)^\beta b^\alpha(t)\, dt = \infty. \tag{21}$$

Proof. Since $\beta = 1 - \alpha$, the first equality in (21) follows. By Lemma 1, any solution x of (1) can be represented in the form (6). Substituting g into (7), we have

$$\varphi'(t) = a^{-\frac{1}{p-1}}(t) B^\alpha(t)\left[F^{\beta_2}(t) - \frac{1}{q} g(t) w\big(\varphi(t)\big)\Phi_p\big(w'\big(\varphi(t)\big)\big)\right]. \tag{22}$$

According to (17), the function g is bounded and (14), (18) yield

$$\left|g(t) w\big(\varphi(t)\big)\Phi_p\big(w'\big(\varphi(t)\big)\big)\right| \leqslant M_1. \tag{23}$$

Let (21) hold. By Lemma 4 there exists a solution $\bar{x}$ of (1) and a constant $k > M_1$ such that $F^{\beta_2}(t) \geqslant k$. Therefore, by integrating (22), we obtain

$$\varphi(t) - \varphi(0) \geqslant (k - M_1)\int_0^t a^{-\frac{1}{p-1}}(\sigma) B^\alpha(\sigma)\, d\sigma.$$

Hence, $\lim_{t\to\infty}\varphi(t) = \infty$ and by Theorem 1, $\bar{x}$ is an oscillatory solution of (1). Conversely, assume

$$\int_0^\infty a^{-\frac{1}{p-1}}(t) B^\alpha(t)\, dt < \infty. \tag{24}$$

According to Lemma 4, for any solution of (1), the function F^{β_2} is bounded. From here, (23), (24), by integrating (22), we obtain that φ is bounded on $\mathbb{R}_+$ for any solution of (1). Hence, by Theorem 1, Eq. (1) is nonoscillatory. □

When (17) holds, an equivalent condition to (21) is given by the following.

Lemma 6. *Assume* (17). *Then* (21) *is equivalent to*

$$\int_0^\infty \bigl(a^{-\frac{1}{p-1}}(t)+b(t)\bigr)\,dt=\infty. \tag{25}$$

Proof. Assume (21). By the Hölder inequality, we have

$$\int_0^s \bigl(a^{-\frac{1}{p-1}}(t)\bigr)^\beta b^\alpha(t)\,dt \leqslant \left(\int_0^s a^{-\frac{1}{p-1}}(t)\,dt\right)^\beta \left(\int_0^s b(t)\,dt\right)^\alpha,$$

and so (25) holds. Conversely, let us show that (25) yields (21). Assume we have $\int_0^\infty a^{-\frac{1}{p-1}}(t)\,dt=\infty$. As already claimed, because g is of the bounded variation, we have $\lim_{t\to\infty} g(t)=\ell_g$, $\ell_g\in\mathbb{R}$. Let $\ell_g<0$. Then there exists $t_0\geqslant 0$ such that $g(t)\leqslant \ell_g/2$ for $t\geqslant t_0$. We have

$$B^{-\alpha}(t)-B^{-\alpha}(t_0)=\alpha\int_{t_0}^t a^{-\frac{1}{p-1}}(\sigma)g(\sigma)\,d\sigma \leqslant \frac{\ell_g\alpha}{2}\int_{t_0}^t a^{-\frac{1}{p-1}}(\sigma)\,d\sigma$$

and, as $t\to\infty$, we obtain a contradiction with the positiveness of B.

Let $\ell_g>0$. Then B is decreasing for large t and $\lim_{t\to\infty}B(t)=\ell_B\geqslant 0$. Then there exists t_1 such that $\ell_B\leqslant B(t)$ and $2\ell_g\geqslant g(t)\geqslant \ell_g/2$ on $[t_1,\infty)$. Assume $\ell_B>0$. We have

$$\begin{aligned}\ell_B-B(t_1)&=\int_{t_1}^\infty B'(t)\,dt=-\int_{t_1}^\infty g(t)B^{1+\alpha}(t)a^{-\frac{1}{p-1}}(t)\,dt\\ &\leqslant -\frac{\ell_g}{2}(\ell_B)^{\alpha+1}\int_{t_1}^\infty a^{-\frac{1}{p-1}}(t)\,dt=-\infty,\end{aligned}$$

i.e., a contradiction. Hence $\ell_B=0$. From this and

$$\begin{aligned}-\ln\frac{B(t)}{B(t_1)}&=-\int_{t_1}^t \frac{B'(s)}{B(s)}\,ds=\int_{t_1}^t g(s)a^{-\frac{1}{p-1}}(s)B^\alpha(s)\,ds\\ &<2\ell_g\int_{t_1}^t a^{-\frac{1}{p-1}}(s)B^\alpha(s)\,ds,\end{aligned}$$

as $t\to\infty$, we obtain the condition (21).

Finally, let $\ell_g=0$. Using the change of the variable $s=s(t)=\int_0^t a^{-1/(p-1)}(\sigma)\,d\sigma$, and denoting by $t(s)$ the inverse function of $s(t)$, we have

$$\frac{d}{ds}B^{-\alpha}\bigl(t(s)\bigr)=-\alpha B^{-(1+\alpha)}\bigl(t(s)\bigr)\frac{d}{ds}B\bigl(t(s)\bigr)=\alpha g\bigl(t(s)\bigr)$$

or, because $\ell_g = 0$,

$$\lim_{s\to\infty} \frac{d}{ds} \frac{1}{B^\alpha(t(s))} = \lim_{s\to\infty} g\bigl(t(s)\bigr) = 0.$$

Hence, by using the generalized l'Hopital rule, we obtain

$$\lim_{s\to\infty} \frac{1}{sB^\alpha(t(s))} = \lim_{s\to\infty} \frac{B^{-\alpha}(t(s))}{s} = 0,$$

i.e., $\lim_{s\to\infty} sB^\alpha(t(s)) = \infty$. Hence $\int_0^\infty B^\alpha(t(s))\,ds = \infty$ and (21) follows by taking into account that

$$\int_0^\infty B^\alpha\bigl(t(s)\bigr)\,ds = \int_0^\infty a^{-\frac{1}{p-1}}(t)B^\alpha(t)\,dt = \infty.$$

Now assume $\int_0^\infty b(t)\,dt = \infty$. We use the so called reciprocity principle (see, e.g., [5]) consisting in the fact that the quasiderivative $z = x^{[1]}$ of a solution x of (1) is a solution of the reciprocal equation of (1)

$$\bigl(b^{\frac{1}{1-q}}(t)\Phi_{q^*}(z')\bigr)' + a^{\frac{1}{1-p}}(t)\Phi_{p^*}(z) = 0, \tag{26}$$

where p^* and q^* denotes the conjugate number of p and q, i.e., $p^* = p/(p-1)$ and $q^* = q/(q-1)$, respectively. This equation comes from (1) when a is replaced by $b^{1/(1-q)}$ and b by $a^{1/(1-p)}$. Denoting by $\bar\alpha$, $\bar B$ and $\bar g$ the above defined expressions α, B and g for (26), we have

$$\bar\alpha = \frac{q(p-1)}{pq-q+p} = \beta, \qquad \bar B(t) = a^{\frac{-1}{p-1}}(t)b^{-1}(t) = 1/B(t),$$
$$\bar g(t) = a^{\frac{1}{p-1}}(t)B'(t)B^{-\alpha-1}(t) = -g(t).$$

Hence, the condition (17) is satisfied for Eq. (26), and we can apply the previous result to (26). We have

$$\int_0^\infty b(t)\bigl(\bar B(t)\bigr)^{\bar\alpha}\,dt = \int_0^\infty b(t)\bigl(B(t)\bigr)^{-\beta}\,dt = \int_0^\infty a^{-\frac{1}{p-1}}(t)B^\alpha(t)\,dt = \infty$$

and the proof is complete. □

From Lemmas 5 and 6 we obtain our main result.

Theorem 2. *Let* (17) *hold and $p \neq q$. Then,* (1) *has an oscillatory solution if and only if*

$$\int_0^\infty \bigl(a^{-\frac{1}{p-1}}(t) + b(t)\bigr)\,dt = \infty.$$

4. Applications

In this section, we will give conditions under which (1) has simultaneously oscillatory and nonoscillatory solutions. The existence of nonoscillatory solutions is given in the following two lemmas.

Lemma 7. *Let* (17) *hold and* $p < q$. *Then* (1) *has a nonoscillatory solution if one of the following conditions* (i) *or* (ii) *is satisfied*:

$$\text{(i)} \quad \int_0^\infty a^{-\frac{1}{p-1}}(t) = \infty, \qquad \int_0^\infty a^{-\frac{1}{p-1}}(t)\left(\int_t^\infty b(s)\,ds\right)^{\frac{1}{p-1}} dt < \infty, \tag{27}$$

$$\text{(ii)} \quad \int_0^\infty b(t) = \infty, \qquad \int_0^\infty b(t)\left(\int_t^\infty a^{-\frac{1}{p-1}}(s)\,ds\right)^{q-1} dt < \infty. \tag{28}$$

Proof. The assertion follows from [9, Theorem 11.4, Corollary 6.1]. □

Lemma 8. *Let* (17) *hold and* $q < p$. *Then* (1) *has a nonoscillatory solution if one of the following conditions* (i) *or* (ii) *is satisfied*:

$$\text{(i)} \quad \int_0^\infty a^{-\frac{1}{p-1}}(t)\,dt = \infty, \qquad \int_0^\infty b(t)\left(\int_0^t a^{-\frac{1}{p-1}}(s)\,ds\right)^{q-1} dt < \infty, \tag{29}$$

$$\text{(ii)} \quad \int_0^\infty b(t)\,dt = \infty, \qquad \int_0^\infty a^{-\frac{1}{p-1}}(t)\left(\int_0^t b(s)\,ds\right)^{\frac{1}{p-1}} dt < \infty. \tag{30}$$

Proof. The assertion follows from [9, Theorem 11.3, Corollary 6.1]. □

From Lemmas 7, 8 and Theorem 2, we obtain the following coexistence result.

Theorem 3. *Let* (17) *holds. Equation* (1) *has oscillatory and nonoscillatory solutions if either* $p < q$ *and one of the conditions* (27), (28) *is satisfied, or* $p > q$ *and one of the conditions* (29), (30) *is satisfied.*

We conclude the paper with a comparison of our results with existing ones in the literature. We start with two examples which illustrate that [9, Theorem 13.1] cannot be applied.

Example 1. Consider the equation

$$\left(|x'|^{p-1}\operatorname{sgn} x'\right)' + t^{-1/\alpha}\left(1 + t^{-\varepsilon}\right)|x|^{q-1}\operatorname{sgn} x = 0, \quad t \in [1, \infty), \tag{31}$$

where $p > 1$, $q > 1$, $\varepsilon > 0$ and α is defined in (3). Then $\int_0^\infty a^{-\frac{1}{p-1}}(t)\,dt = \infty$, $\int_0^\infty b(t)\,dt < \infty$ and

$$g(t) = \alpha^{-1}\left(1 + t^{-\varepsilon}\right)^{-\alpha} + \varepsilon t^{-\varepsilon}\left(1 + t^{-\varepsilon}\right)^{-1-\alpha}.$$

Hence, $\int_0^\infty |g'(t)|\,dt < \infty$ and (17) holds. By Theorem 1, Eq. (31) has an oscillatory solution. Observe that the function

$$A_1(t) = b(t)a^{\frac{1}{p-1}}(t)\left(\int_0^t a^{-\frac{1}{p-1}}(s)\,ds\right)^{1/\alpha} = 1 + t^{-\varepsilon}$$

is decreasing on $[1,\infty)$, while Theorem 13.1 of [9] requires that A_1 is nondecreasing and so it cannot be applied.

Example 2. Consider the equation

$$\left(\left(t^{1/\beta}\left(1-t^{-\varepsilon}\right)\right)^{p-1}|x'|^{p-1}\operatorname{sgn} x'\right)' + |x|^{q-1}\operatorname{sgn} x = 0, \quad t \in [1,\infty), \tag{32}$$

where $1 < p < q$, $\varepsilon > 0$ and β is given in (3). Then $\int_0^\infty a^{-\frac{1}{p-1}}(t)\,dt < \infty$ and

$$g(t) = -\beta^{-1}\left(1-t^{-\varepsilon}\right)^{\beta} - \varepsilon\left(1-t^{-\varepsilon}\right)^{\beta-1}t^{-\varepsilon}.$$

Hence, $\int_0^\infty |g'(t)|\,dt < \infty$ and (17) holds. In addition, (28) is satisfied and therefore (32) has oscillatory and nonoscillatory solutions. Note that [9, Theorem 13.1] cannot be applied because the function

$$A_2(t) = \frac{1}{b(t)a^{\frac{1}{p-1}}(t)}\left(\int_0^t b(s)\,ds\right)^{1/\beta} = \frac{1}{1-t^{-\varepsilon}}$$

is decreasing on $[1,\infty)$ and Theorem 13.1 requires that A_2 is nondecreasing.

Finally, we discuss assumptions in Theorem 2 with those ones in [10] for the equation

$$x'' + b(t)|x|^{\mu}\operatorname{sgn} x = 0, \quad \mu > 0,\ \mu \neq 1. \tag{33}$$

Clearly (25) is satisfied, and (17) reads for (33) as follows: $b \in AC^1_{\mathrm{loc}}(\mathbb{R}_+)$ and the function

$$g(t) = -b'(t)\left[b(t)\right]^{-(\mu+5)/(\mu+3)} \quad \text{is of the bounded variation.} \tag{34}$$

Sufficient conditions ensuring the existence of oscillatory solutions given in [10] are the following: $b \in AC(\mathbb{R}_+)$, there exists $h > 0$ such that

$$\Psi(t) = b(t)t^{(\mu+3)/2} \geqslant h, \quad t \geqslant t_0 > 0, \tag{35}$$

and

$$\int^{\infty} \Psi'_-(t)\,dt < \infty, \quad \text{where } \Psi'_-(t) = -\min\left\{\Psi'(t), 0\right\}. \tag{36}$$

It seems rather difficult to compare the condition (34) with (36). To illustrate the situation, consider Eq. (33) on $[1,\infty)$ with

$$b(t) = t^{-\frac{\mu+3}{2}}\Psi(t), \quad \Psi(t) = t\left(\frac{6}{5} + \sin\ln t\right).$$

Then (36) is not satisfied, i.e., the result in [10] cannot be applied. The condition (34) is satisfied for large μ, thus, by Theorem 2, such equation has an oscillatory solution. Moreover, because (27) holds for $\mu > 3$, we can apply Theorem 3 to obtain the coexistence of oscillatory and nonoscillatory solutions.

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